

SIMPLE ESTIMATES FOR ELLIPSOID MEASURES

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ABSTRACT. We write down estimates for the surface area, and more generally, integral mean curvatures of an ellipsoid E in \mathbb{E}^n in terms of the lengths of the major semi-axes. We give applications to estimating the area of parallel surfaces and volume of the tubular neighborhood of E , to the counting of lattice points contained in E and to estimating the shape of the John ellipsoid of a convex body K .

INTRODUCTION

Consider an ellipsoid $E \subset \mathbb{E}^n$. Such an ellipsoid is defined as the set

$$(1) \quad E = \{x \in \mathbb{E}^n \mid \|Ax\| = 1\},$$

where A is a non-singular linear transformation of \mathbb{E}^n . Remark that

$$\|Ax\| = \langle Ax, Ax \rangle = \langle x, A^t Ax \rangle.$$

The matrix $Q = A^t A$ is a positive definite matrix, with eigenvalues $\lambda_1, \dots, \lambda_n$, whose (positive) square roots $\sigma_1, \dots, \sigma_n$ are the so-called *singular values* of A . Their geometric significance is that the *major semi-axes* of E are the quantities $a_i = 1/\sigma_i$. After an orthogonal change of coordinates, we can write

$$(2) \quad E = \{x \in \mathbb{E}^n \mid \sum_{i=1}^n x_i^2 \lambda_i = \sum_{i=1}^n x_i^2 \sigma_i^2 = \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1\}.$$

It is evident from (1) and (2) that the volume of E is given by

$$(3) \quad \text{vol } E = \frac{\kappa_n}{\det A} = \kappa_n \prod_{i=1}^n a_i,$$

where κ_n is the volume of the unit ball in \mathbb{E}^n . The formula (3) is deceptive, in that, as is quite well known, there is no simple expression (in terms of the major semi-axis lengths) for the *surface area* of E ,

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even in 2 dimensions. In this note we write down an estimate for the surface area and *integral mean curvatures of E* in terms of the major semi-axes. These estimate differ from the true values by a factor which depends only on the dimension n . It should be noted that the estimate on the dimensional factor is extremely crude, but the methods used to get sharper results are quite different from those used in this note, and so we postpone that to a different note [2]. In any event, these estimate also allow us to estimate the volume of the set

$$E_\rho = \{x \mid d(x, E) \leq \rho\},$$

as well as of the surface area of ∂E_ρ . Despite appearances, E_ρ is usually *not* an ellipsoid.

Our estimates for the integral mean curvatures (which are defined in the next section of this note) will have the following form:

$$(4) \quad c_{n,i} \leq \frac{\mathcal{M}_i(\partial E)}{s_{n-1-i}(a_1, \dots, a_n)} \leq C_{n,i},$$

where $C_{n,i}/c_{n,i} \leq n^{(n-i)/2}$, and s_k is the k -th elementary symmetric function, defined by

$$\prod_{i=1}^n (x + a_i) = \sum_{k=0}^n x^k s_{n-k}(a_1, \dots, a_n).$$

It should be noted that \mathcal{M}_0 is the surface area of ∂E . The estimates for $\text{vol } E_\rho$ and $\text{vol}_{n-1} \partial E_\rho$ are given in Theorem 3.

1. INTEGRAL GEOMETRY

The reference for the results recalled in this section is Santalo's book [3]. Let K be a convex body in \mathbb{E}^n ; we will initially assume that ∂K is of smoothness at least C^2 . The k -th *mean curvature* $m_k(\partial K, x)$ is defined as

$$\binom{n-1}{k} m_k(x) = s_k(k_1(x), \dots, k_{n-1}(x)),$$

where k_1, \dots, k_{n-1} are the *principal curvatures* of ∂K at the point x . The k -th *integral mean curvature* \mathcal{M}_k is defined as

$$\mathcal{M}_k = \int_{\partial K} m_k(x) dA,$$

where dA is the usual surface measure on ∂K . In particular, since $m_0 = 1$, it follows that $\mathcal{M}_0 = \text{vol}_{n-1} \partial K$.

The properties of \mathcal{M}_k we will use are:

(1) The area of ∂K_ρ is given by ([3, (13.43)]:

$$(5) \quad \text{vol}_{n-1} \partial K_\rho = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathcal{M}_k(\partial K) \rho^k,$$

while ([3, (13.44)])

$$(6) \quad \text{vol } K\rho = \text{vol } K + \int_0^\rho \text{vol}_{n-1} K_\tau d\tau.$$

(2) Let $G_{n,r}$ be the Grassmanian of affine r -planes in \mathbb{E}^n , with a suitably normalized invariant measure μ . Then ([3, (14.1)])

$$(7) \quad \mu(\{L_r \in G_{n,r} \mid L_r \cap K \neq \emptyset\}) = \frac{\omega_{n-2} \cdots \omega_{n-r-1}}{(n-r)\omega_{r-1} \cdots \omega_0} \mathcal{M}_{r-1}(\partial K),$$

where ω_k is the surface area of the unit sphere in \mathbb{E}^{k+1} .

The above relationships can be used to define integral mean curvatures for not-necessarily- C^2 convex bodies, and from now on the assumption of regularity will be dropped. An important corollary of (7) is

Theorem 1 (Archimedes' axiom). *$\partial \mathcal{K}$ is monotonic under inclusion. That is, if $K_1 \subseteq K_2$, then $\mathcal{M}_i(\partial K_1) \leq \mathcal{M}_i(\partial K_2)$. The inequality is strict if $K_2 \setminus K_1$ has non-empty interior, and $i < n-1$, where n is the dimension of the ambient Euclidean space.*

Remark 2. I use the name ‘‘Archimedean axiom’’ because Archimedes needed a result like this in the plane in order to make rigorous his computation of the arclengths of curves (the circle, for example). Archimedes was unable to prove this result from first principles so he postulated it as an axiom.

2. POLYHEDRA

The formula (5) can be used to compute integral mean curvatures for polytopes. To wit, let P be a convex polytope in \mathbb{E}^n .

$$(8) \quad \binom{n-1}{i} \mathcal{M}_i(\partial P) = \sum_{\text{codim. } i \text{ faces } f \text{ of } \partial P} \text{vol}_{n-i-1} f \alpha^*(f),$$

where $\alpha^*(f)$ is the *exterior angle* at f , described as follows: Consider the Gauss map, which maps each point p of ∂P to the set of outer normals to the support planes to P passing through p . The image of all of ∂P will be the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{E}^n$, and the combinatorial structure of ∂P will induce a *dual* cell decomposition C of \mathbb{S}^{n-1} , in

particular, the image of a codimension- i face f of ∂P will be an i -dimensional totally geodesic face f^* of C . The i -dimensional area of that face is the exterior angle at f .

Suppose now that translates of P tile \mathbb{E}^n , so that P is a fundamental domain for a free action of a group G of translations on \mathbb{E}^n . In particular, G acts on ∂P , preserving the combinatorial structure. Let the quotient be ∂P_G . Then

$$(9) \quad \binom{n-1}{i} \mathcal{M}_i = \omega_i \sum_{\text{codimension } i \text{ faces } f \text{ of } \partial P_G} \text{vol}_{n-1-i} f.$$

In particular, if P is a rectangular parallelepiped:

$$P = [0, l_1] \times [0, l_2] \times \cdots \times [0, l_n],$$

then we obtain

$$(10) \quad \mathcal{M}_i(P) = \omega_i s_{n-1-i}(l_1, \dots, l_n).$$

3. BACK TO THE ELLIPSOID

Consider again the ellipsoid

$$E = \left\{ x \mid \sum_{i=1}^n \frac{x_i^2}{a_i^2} = 1 \right\}.$$

The ellipsoid E is *inscribed* into a *box* P : a right parallelepiped whose sides are parallel to the coordinate axes and have lengths $2a_1, \dots, 2a_n$. Consider now the diagonal matrix A whose entries are $A_{ii} = 1/a_i$. $A(E)$ is the unit ball B_n , while $A(P)$ is the cube $[-1, 1]^n$, circumscribed around B_n . It is clear that the cube $[-1/\sqrt{n}, 1/\sqrt{n}]^n$ is inscribed in ∂B_n , and so P/\sqrt{n} is inscribed in E . It follows that

$$(11) \quad \frac{\mathcal{M}_i(P)}{(\sqrt{n})^{n-1-i}} = \mathcal{M}_i\left(\frac{P}{\sqrt{n}}\right) \leq \mathcal{M}_i(E) \leq \mathcal{M}_i(P),$$

or

$$(12) \quad \left(\frac{2}{\sqrt{n}}\right)^{n-1-i} \frac{s_{n-1-i}(l_1, \dots, l_n)}{\binom{n-1}{i}} \leq \mathcal{M}_i(E) \leq 2^{n-1-i} \frac{s_{n-1-i}(l_1, \dots, l_n)}{\binom{n-1}{i}}.$$

The following theorem follows immediately.

Theorem 3. *Let*

$$f(\rho) = \frac{1}{\rho} \left[\prod_{i=1}^n (\rho + 2l_i) - \rho^n - 2^n \prod_{i=1}^n l_i \right].$$

Then the area of the equidistant surface ∂E_ρ is bounded as follows:

$$c_n f(\rho) \leq \text{vol}_{n-1} \partial E_\rho \leq C_n f(\rho),$$

where

$$C_n/c_n \leq (\sqrt{n})^{n-1} \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}.$$

Remark 4. A related estimate for the volume of E_ρ immediately follows from Theorem 3 and Eq. (6).

4. APPLICATIONS AND COMMENTS

4.1. Lattice Points. The question at the root of this note was the following: Consider an ellipsoid E , and consider the number $N(e)$ of points of the integer lattice in E . How do we estimate $\Delta(E) = |N(E) - \text{vol } E|$?

By a standard argument (see, eg, Landau [1]), $\Delta(E)$ is dominated by the volume of a tubular neighborhood of radius \sqrt{n} of ∂E . The results of Theorem 3 and Remark 4, we see that

$$(13) \quad \Delta(E) \leq C_n \int_0^{\sqrt{n}} f(\rho) d\rho \leq C'_n f(\sqrt{n}),$$

where f is defined in the statement of Theorem 3.

4.2. Convex Bodies. It is a well-known theorem of Fritz John that for any convex body K , there exists an ellipsoid E_K , such that $E_K/n \subset K \subset E_K$ – for centrally-symmetric K , E_K/n can be improved to E_K/\sqrt{n} . The results above allow us to estimate the symmetric functions of the semi-axes (and hence the semi-axes themselves) of the John ellipsoid in terms of the mean curvature integrals of K , and *vice versa*.

4.3. Improvements. In the paper [2], among other results, the dimensional constant for the surface area of an ellipsoid is tightened quite considerably (from roughly $n^{n/2}$ to roughly n).

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